AD-A104 187 UTAH UNIV SALT LAKE CITY DEPT OF MATHEMATICS F/G 20/1
TRANSIENT ACOUSTIC WAVE PROPAGATION IN STRATIFIED FLUIDS.(U)
JUL 81 C H WILCOX N00014-76-C-0276
UNCLASSIFIED TSR-42 NL

1 OF 1
AD A
104187

END
DATE
FILMED
10-81
DTIC

Department of Mathematics

UNIVERSITY OF UTAH

Salt Lake City, Utah
U. S. A.

9

DTIC
ELECTE
SEP 1 4 1981
A

Transient Acoustic Wave Propagation

in Stratified Fluids

C. H. Wilcox

Technical Summary Report #42

July 1981

Prepared under Contract No. N00014-76-C-0276

Task No. NR-041-370

for

Office of Naval Research

Contents

Abstract.

Transient acoustic wave propagation is analyzed for the case of plane-stratified fluids having density $\rho(y)$ and sound speed $c(y)$ at depth y. For infinite fluids it is assumed that the (in general discontinuous) functions $\rho(y)$, $c(y)$ are uniformly positive and bounded and satisfy

$$|\rho(y) - \rho(\pm\infty)| \le C(\pm y)^{-\alpha}, \quad |c(y) - c(\pm\infty)| \le C(\pm y)^{-\alpha}$$

for $\pm y > 0$, where $\alpha > 3/2$. Semi-infinite and finite layers are also treated. The acoustic potential is a solution of the wave equation

$$\frac{\partial^2 u}{\partial t^2} - c^2(y)\, \rho(y)\, \nabla \cdot (\rho^{-1}(y)\, \nabla u) = f(t,x,y)$$

where $x = (x_1, x_2)$ are horizontal coordinates and $f(t,x,y)$ characterizes the wave sources. The principal results of the analysis show that u is the sum of a free component, which behaves like a diverging spherical wave for large t, and a guided component which is approximately localized in regions $|y - y_j| < h_j$ where $c(y)$ has minima and propagates outward in horizontal planes like a diverging cylindrical wave.

§1. Introduction.

This report presents an analysis of the structure of transient acoustic waves in stratified fluids whose densities and sound speeds are functions of a single depth coordinate. The acoustic field in such a fluid may be described by a real-valued function $u(t,x,y)$ (the acoustic potential or the excess pressure) that satisfies the wave equation

$$(1.1) \qquad \frac{\partial^2 u}{\partial t^2} - c^2(y)\ \rho(y)\ \nabla \cdot (\rho^{-1}(y) \nabla u) = f(t,x,y)$$

where t is a time coordinate, y is a depth coordinate, $x = (x_1,x_2)$ are Cartesian coordinates in a horizontal plane, $\nabla = (\partial/\partial x_1, \partial/\partial x_2, \partial/\partial y)$ and $f(t,x,y)$ is a function that characterizes the wave sources. $c(y)$ and $\rho(y)$ are the variable sound speed and density, respectively, and $\rho^{-1}(y) = 1/\rho(y)$.

This report is a sequel to the author's report [12] on "Spectral Analysis of Sound Propagation in Stratified Fluids." That work contains a spectral analysis of the acoustic propagator

$$(1.2) \qquad Au = -c^2(y)\ \rho(y)\ \nabla \cdot (\rho^{-1}(y) \nabla u)$$

for the cases of an unlimited fluid ($-\infty < x_1,x_2,y < \infty$), a semi-infinite layer ($-\infty < x_1,x_2 < \infty$ and $0 < y < \infty$) and a finite layer ($-\infty < x_1,x_2 < \infty$ and $0 < y < h < \infty$). The integration of (1.1) below is based on the spectral analysis of [12] and the notation and results of [12] are used throughout this report. As in [12], only the case of an unlimited fluid is presented in detail. The modifications required in the second and third cases are described in §6 at the end of the report.

The analysis of the structure of transient acoustic waves in plane stratified media was initiated by the author in 1973 [7]. This work dealt with the special case of the Pekeris profile (propagation in a half-space, $\rho(y) = \text{const.}$, $c(y)$ piecewise constant). In 1974 the results were extended to the symmetric Epstein profile ($\rho(y) = \text{const.}$, $c^{-2}(y) = K \operatorname{sech}^2 (y/H) + M$) [8]. The general Epstein profile ($\rho(y) = \text{const.}$, $c^{-2}(y) = K \operatorname{sech}^2 (y/H) + L \tanh (y/H) + M$) was treated in 1979 [11] using the spectral analysis of the Epstein operator due to Guillot and Wilcox [2, 3]. A preliminary version of [11] was announced in 1978 [1]. The analysis of these cases was based on explicit representations of their normal mode functions by means of well-known special functions. Extension of the analysis to larger classes of stratified fluids had to await the extension of the spectral analysis of the acoustic propagators (1.2) to such classes.

The class of stratified fluids treated in [12] was characterized by the properties

$$\rho(y) \text{ and } c(y) \text{ are Lebesgue measurable,} \tag{1.3}$$

$$0 < \rho_m \le \rho(y) \le \rho_M < \infty, \quad 0 < c_m \le c(y) \le c_M < \infty, \tag{1.4}$$

$$\pm\int_0^{\pm\infty} |\rho(y) - \rho(\pm\infty)|\, dy < \infty, \quad \pm\int_0^{\pm\infty} |c(y) - c(\pm\infty)|\, dy < \infty, \tag{1.5}$$

where ρ_m, ρ_M, $\rho(\pm\infty)$, c_m, c_M and $c(\pm\infty)$ are constants. In this report the results of [10] and [11] on transient acoustic waves in stratified fluids are extended to the class of fluids characterized by (1.3), (1.4) and

$$\left\{\begin{array}{l} |\rho(y) - \rho(\pm\infty)| \le C(\pm y)^{-\alpha} \\ \\ |c(y) - c(\pm\infty)| \le C(\pm y)^{-\alpha} \end{array}\right\} \text{ for } \pm y > 0, \tag{1.6}$$

where C and α are constants and

$$\alpha > \frac{3}{2} . \tag{1.7}$$

It is clear that (1.6) implies (1.5). It will be seen from the analysis below that (1.6) could be replaced by other order conditions at $y = \pm\infty$. It is not known whether the results of this report hold for the entire class of luids defined by (1.3), (1.4), (1.5).

The remainder of this report is organized as follows. In §2 the normal mode expansions for unlimited fluids of [12] are used to decompose transient acoustic fields into free and guided components and to obtain integral representations of these components. The behavior for large t of the free and guided components is derived in §3 and §4, respectively. In §5 these results are used to calculate the asymptotic distribution for large t of the wave energy. §6 presents corresponding results for semi-infinite and finite layers.

BLANK PAGE

§2. Normal Mode Expansions of Transient Acoustic Fields.

The integration of the wave equation (1.1) will be based on the spectral analysis of the acoustic propagator A in the Hilbert space $\mathcal{H} = L_2(R^3, c^{-2}(y)\, \rho^{-1}(y)\, dxdy)$, as developed in [12]. To this end (1.1) is interpreted as the equation

$$(2.1) \qquad \frac{d^2u}{dt^2} + Au = f(t,\cdot), \quad t \in R,$$

for a function $t \to u(t,\cdot) \in \mathcal{H}$. The wave sources will be assumed to act during a time interval $[0,T]$, so that supp $f \subset [0,T]$. The corresponding acoustic wave is the solution of (2.1) that satisfies the initial condition

$$(2.2) \qquad u(t,\cdot) = 0 \text{ for all } t < 0.$$

The solution, based on the spectral theorem for the selfadjoint operator $A \geq 0$, is given by Duhamel's integral

$$(2.3) \qquad u(t,\cdot) = \int_0^t \{A^{-1/2} \sin (t - \tau)A^{1/2}\}\ f(\tau,\cdot)d\tau, \ t \geq 0.$$

Indeed, if $f \in C([0,T],\mathcal{H})$ then (2.3) is the unique "solution with finite energy" of [6], while if $f \in C([0,T],D(A^{1/2}))$ then (2.3) is the "strict solution with finite energy." In addition, if $f \in C([0,T],D(A^{-1/2}))$ then

$$(2.4) \qquad u(t,x,y) = \text{Re}\ \{v(t,x,y)\}$$

where $v(t,\cdot)$ is the complex-valued potential defined by

$$(2.5) \qquad v(t,\cdot) = i \exp\{-i t A^{1/2}\} A^{-1/2} \int_0^t \exp\{i \tau A^{1/2}\} f(\tau,\cdot) d\tau .$$

In particular,

$$(2.6) \qquad v(t,\cdot) = \exp\{-i t A^{1/2}\} h \text{ for all } t \geq T$$

where

$$(2.7) \qquad h = i A^{-1/2} \int_0^T \exp\{i \tau A^{1/2}\} f(\tau,\cdot) d\tau .$$

The initial value problem

$$(2.8) \qquad \frac{d^2u}{dt^2} + Au = 0 \text{ for } t > 0,$$

$$(2.9) \qquad u(0) = f, \ \frac{du(0)}{dt} = g$$

can be treated by the same formalism. Indeed, if $f \in D(A^{1/2})$ and $g \in D(A^{-1/2})$ then the solution of (2.8), (2.9) is given by (2.4), (2.6) with $h = f + i A^{-1/2} g \in D(A^{1/2})$; cf. [9, Ch. 3].

The integral

$$(2.10) \qquad E(u,K,t) = \int_K \{ |\nabla u|^2 + c^{-2}(y) \left(\frac{\partial u}{\partial t}\right)^2 \} \rho^{-1}(y) \, dxdy$$

may be interpreted as the energy of the acoustic field u in the set $K \subset R^3$ at time t. Moreover, A is the selfadjoint operator in $\mathcal{H}$ associated with the sesquilinear form A on $\mathcal{H}$ defined by $D(A) = L_2^1(R^3) \subset \mathcal{H}$ and

$$A(u,v) = \int_{R^3} \nabla\bar{u} \cdot \nabla v \, \rho^{-1}(y) \, dxdy \tag{2.11}$$

It follows from Kato's second representation theorem [4, p. 331] that $D(A^{1/2}) = L_2^1(R^3)$ and for all $u \in D(A^{1/2})$

$$\|A^{1/2} u\|_{\mathcal{H}}^2 = A(u,u) = \int_{R^3} |\nabla u|^2 \rho^{-1}(y) \, dxdy. \tag{2.12}$$

Hence the total energy satisfies

$$E(u,R^3,t) = \|A^{1/2} u\|_{\mathcal{H}}^2 + \left\|\frac{\partial u}{\partial t}\right\|_{\mathcal{H}}^2. \tag{2.13}$$

If $h \in D(A^{1/2})$ and u is defined by (2.4), (2.6) then a simple calculation shows that

$$E(u,R^3,t) = \|A^{1/2} h\|_{\mathcal{H}}^2 \text{ for all } t \geq T. \tag{2.14}$$

The analysis of the structure of the acoustic potential (2.4), (2.6) presented below is based on the normal mode expansion for A of [12, §8]. The orthogonal projections $\{P_+, P_-, P_0, P_1, P_2, \cdots\}$ in $\mathcal{H}$ defined by the normal modes form a complete family that reduces A [12, Corollaries 8.4 and 8.10]. Hence the same is true of the orthogonal projections

$$P_f = P_+ + P_- + P_0 \tag{2.15}$$

and

$$P_g = \sum_{k=1}^{N_0-1} P_k. \tag{2.16}$$

This provides a decomposition

$$u(t,\cdot) = u_f(t,\cdot) + u_g(t,\cdot) \tag{2.17}$$

into orthogonal partial waves

$$u_f(t,\cdot) = P_f\, u(t,\cdot),\ u_g(t,\cdot) = P_g\, u(t,\cdot). \tag{2.18}$$

The *function* u_f, called the free component of u, will be shown to behave for large times like a diverging spherical wave. The function u_g, called the guided component of u, will be shown to be approximately localized near one or more of the planes y = const. where c(y) has local minima and to propagate outward in horizontal planes like a diverging cylindrical wave. This second component shows the profound effect of acoustic ducts on transient acoustic waves. It is absent when c(y) has no local minima.

§3. Transient Free Waves.

The normal mode expansions of [12] are used in this section to calculate the asymptotic behavior for $t \to \infty$ of the free component $u_f(t,\cdot) = P_f\, u(t,\cdot)$. The principal result is that in each of the half-spaces $R_+^3(d)$ and $R_-^3(d)$, where $R_\pm^3(d) = \{(x,y) : \pm(y - d) > 0\}$, $u_f(t,\cdot)$ is asymptotically equal to a wave function for a homogeneous fluid with propagation speed $c(\infty)$ and $c(-\infty)$, respectively. It is this behavior that motivates the term "free component" for $u_f(t,\cdot)$.

It will be assumed that the total acoustic potential u satisfies $u(t,\cdot) = \mathrm{Re}\,\{v(t,\cdot)\}$ where $v(t,\cdot) = \exp\{-i\,t\,A^{1/2}\}\,h$ and $h \in D(A^{1/2})$ (see §2). The corresponding partial waves $u_k(t,\cdot) = P_k\, u(t,\cdot)$ with $k \geq 1$ satisfy $u_k(t,\cdot) = \mathrm{Re}\,\{\exp\,(-i\,t\,A^{1/2})\,P_k h\}$. This follows from the fact that the normal mode functions $\psi_k(y,p)$ are real for $k \geq 1$, which implies that $P_k(\overline{h}) = \overline{P_k(h)}$. It follows by addition that $u_g(t,\cdot) =$ $\mathrm{Re}\,\{\exp\,(-i\,t\,A^{1/2})\,P_g h\} = \mathrm{Re}\,\{v_g(t,\cdot)\}$ and hence

$$(3.1)\qquad \begin{cases} u_f(t,\cdot) = \mathrm{Re}\,\{v_f(t,\cdot)\}, \\ \\ v_f(t,\cdot) = \exp\,(-i\,t\,A^{1/2})\,P_f h = \exp\,(-i\,t\,A^{1/2})\,h_f. \end{cases}$$

The normal mode representations

$$(3.2)\qquad v_f(t,x,y) = \int_{R^3} \phi_\pm(x,y,p,q)\,\exp\{-it\lambda^{1/2}(p,q)\}\,\hat{h}_\pm(p,q)\,dpdq$$

of [12, §8] provide the starting point for calculating the asymptotic behavior of $u_f(t,x,y)$ for large t. The integral in (3.2) will, in general, converge only in $\mathcal{H}$. For brevity the $\mathcal{H}$-lim notation of [12] is suppressed in this report.

Equation (3.2) gives two representations of v_f corresponding to the two families ϕ_+ and ϕ_-. The calculations below are based on the ϕ_--representation which has been found to yield the simplest form of the asymptotic wave function. It will be convenient to introduce the characteristic functions χ_+, χ_0 and χ_- of the cones C_+, C_0 and C_- in (p,q)-space [12, (1.48)-(1.51)] and to decompose $\hat{h}_-$ as

(3.3) $$\hat{h}_-(p,q) = \ell(p,q) + m(p,q) + n(p,q)$$

where $\ell = \chi_+ \hat{h}_-$, $m = \chi_0 \hat{h}_-$ and $n = \chi_- \hat{h}_-$. The corresponding decomposition of v_f is

(3.4) $$v_f = v_\ell + v_m + v_n$$

where

(3.5) $$\begin{cases} v_\ell = \exp\,(-i\,t\,A^{1/2})\ \Phi_-^* \ell \\ v_m = \exp\,(-i\,t\,A^{1/2})\ \Phi_-^* m \\ v_n = \exp\,(-i\,t\,A^{1/2})\ \Phi_-^* n \end{cases}$$

The behavior for $t \to \infty$ of these three functions will be analyzed separately.

<u>Behavior of v_ℓ.</u> The partial wave v_ℓ has the representation

(3.6) $$v_\ell(t,x,y) = \int_{C_+} \phi_-(x,y,p,q)\ \exp(-i\,t\,\omega_+(p,q))\ \ell(p,q)\ dpdq$$

where

(3.7) $$\omega_\pm(p,q) = c(\pm\infty)\ \sqrt{|p|^2 + q^2}.$$

(Recall $\lambda(p,q) = \omega_{\pm}^2(p,q)$ for $\pm q > 0$.) To discover the behavior of $v_{\ell}(t,x,y)$ for $(x,y) \in R_{\pm}^2(d)$ and $t \to \infty$ it will be convenient to write $\phi_-(x,y,p,q)$ in a way that puts in evidence its behavior for $y \to \pm\infty$. To this end recall that by [12, (1.53)-(1.60)]

$$\phi_-(x,y,p,q) = (2\pi)^{-1}\, c(\infty)\,(2q)^{1/2}\, e^{ip\cdot x}\, \overline{\psi_+(y,|p|,\lambda)} \tag{3.8}$$

for $(p,q) = X_+(p,\lambda) \in C_+$ (and hence $\lambda = \lambda(p,q) = c^2(\infty)(|p|^2 + q^2)$). Moreover, by [12, (4.5), (4.6) and (8.41)] one can write

$$\begin{aligned} \psi_+(y,\mu,\lambda) &= \left(\frac{\rho(\infty)}{4\pi q_+(\mu,\lambda)}\right)^{1/2} T_+(\mu,\lambda)\ \phi_4(y,\mu,\lambda) \\ &= \left(\frac{\rho(\infty)}{4\pi q_+(\mu,\lambda)}\right)^{1/2} T_+(y,\mu,\lambda)\ \exp\{-i\,y\,q_-(\mu,\lambda)\} \end{aligned} \tag{3.9}$$

where

$$T_+(y,\mu,\lambda) = T_+(\mu,\lambda)\ \phi_4(y,\mu,\lambda)\exp\{i\,y\,q_-(\mu,\lambda)\} \to T_+(\mu,\lambda),\ \ y \to -\infty. \tag{3.10}$$

Similarly, by [12, (4.1) and (4.6)-(4.11)] one can write

$$\begin{aligned} \psi_+(y,\mu,\lambda) = \left(\frac{\rho(\infty)}{4\pi q_+(\mu,\lambda)}\right)^{1/2} & [I_+(y,\mu,\lambda)\ \exp\{-i\,y\,q_+(\mu,\lambda)\} \\ & + R_+(y,\mu,\lambda)\ \exp\{i\,y\,q_+(\mu,\lambda)\}] \end{aligned} \tag{3.11}$$

where

$$\left.\begin{cases} I_+(y,\mu,\lambda) = \phi_2(y,\mu,\lambda)\ \exp\{i\,y\,q_+(\mu,\lambda)\} \to 1 \\ R_+(y,\mu,\lambda) = R_+(\mu,\lambda)\ \phi_1(y,\mu,\lambda)\exp\{-i\,y\,q_+(\mu,\lambda)\} \to R_+(\mu,\lambda) \end{cases}\right\}\ y \to +\infty. \tag{3.12}$$

Combining (3.6), (3.8) and (3.11) gives

$v_\ell(t,x,y)$

$$= c(\infty)\ \rho(\infty)^{1/2}\ (2\pi)^{-3/2} \int_{C_+} \exp\{i(x\cdot p + y\,q - t\omega_+(p,q)\}\ \overline{I_+(y,|p|,\lambda)}\ \ell(p,q)dpdq$$

(3.13)

$$+\ c(\infty)\ \rho(\infty)^{1/2}\ (2\pi)^{-3/2} \int_{C_+} \exp\{i(x\cdot p - y\,q - t\omega_+(p,q)\}\ \overline{R_+(y,|p|,\lambda)}\ell(p,q)dpdq.$$

It is natural to expect that in $R_+^3(d)$ the partial wave $v_\ell(t,x,y)$ will propagate as $t \to \infty$ into regions where y is large and hence $I_+(y,|p|,\lambda)$ and $R_+(y,|p|,\lambda)$ are near their limiting values. Thus the representation (3.13) suggests the conjecture that

$$(3.14) \qquad v_\ell(t,\cdot) \sim v_\ell^0(t,\cdot) + v_\ell^1(t,\cdot) \text{ in } L_2(R_+^3(d)),\ t \to \infty$$

where v_ℓ^0 *and* v_ℓ^1 *are defined by*

$$(3.15)\ v_\ell^0(t,x,y) = c(\infty)\rho(\infty)^{1/2}\ (2\pi)^{-3/2} \int_{C_+} \exp\{i(x\cdot p+y\,q - t\omega_+(p,q)\}\ell(p,q)dpdq$$

and

$$v_\ell^1(t,x,y) = c(\infty)\rho(\infty)^{1/2}\ (2\pi)^{-3/2} \int_{C_+} \exp\{i(x\cdot p-y\,q - t\omega_+(p,q)\} \times$$
$$\times\ \overline{R_+(|p|,\lambda)}\ \ell(p,q)dpdq$$

(3.16)

$$= c(\infty)\rho(\infty)^{1/2}\ (2\pi)^{-3/2} \int_{-C_+} \exp\{i(x\cdot p+y\,q - t\omega_+(p,q)\} \times$$
$$\times\ \overline{R_+(|p|,\lambda)}\ \ell(p,-q)dpdq$$

where $-C_+ = \{(p,q) : (p,-q) \in C_+\} = \{(p,q) : q < -a|p|\}$. Note that v_ℓ^0 and v_ℓ^1 are waves in a homogeneous medium with density $\rho(\infty)$ and sound speed $c(\infty)$. More precisely,

$$(3.17)\quad \begin{cases} v_\ell^0(t,\cdot) = \exp\,(-i\,t\,c(\infty)\,A_0^{1/2})\,h_\ell \\ v_\ell^1(t,\cdot) = \exp\,(-i\,t\,c(\infty)\,A_0^{1/2})\,h_\ell^1 \end{cases}$$

where A_0 is the selfadjoint realization in $L_2(R^3)$ of $-\Delta = -(\partial^2/\partial x_1^2 + \partial^2/\partial x_2^2 + \partial^2/\partial y^2)$ and h_ℓ and h_ℓ^1 are the functions in $L_2(R^3)$ whose Fourier transforms are

$$(3.18)\quad \begin{cases} \hat{h}_\ell(p,q) = c(\infty)\;\rho(\infty)^{1/2}\,\ell(p,q) = c(\infty)\;\rho(\infty)^{1/2}\,\chi_+(p,q)\;\hat{h}_-(p,q), \\ \hat{h}_\ell^1(p,q) = c(\infty)\;\rho(\infty)^{1/2}\,\overline{R_+(|p|,\lambda)}\;\ell(p,-q) \end{cases}$$

$$= c(\infty)\;\rho(\infty)^{1/2}\,\overline{R_+(|p|,\lambda)}\;(1 - \chi_+(p,q))\;\hat{h}_-(p,-q).$$

Both functions are in $L_2(R^3)$ because $\hat{h}_- \in L_2(R^3)$ and $|R_+(|p,\lambda)| \le \rho^{1/2}(\infty)$ by the conservation law [12, (1.46)]. Moreover, supp $\hat{h}_\ell^1 \subset -C_+$ and hence the theory of asymptotic wave functions for d'Alembert's equation [9, Ch. 2] implies that $v_\ell^1(t,\cdot) \sim 0$ in $L_2(R_+^3(d))$ when $t \to \infty$. Combining this with (3.14) gives

$$(3.19)\qquad v_\ell(t,\cdot) \sim v_\ell^0(t,\cdot) \text{ in } L_2(R_+^3(d)),\; t \to \infty.$$

Now consider the behavior of $v_\ell(t,x,y)$ for $(x,y) \in R_-^3(d)$, $t \to \infty$. Combining (3.6), (3.9) and (3.10) gives

$$(3.20)\quad v_\ell(t,x,y) = c(\infty)\;\rho(\infty)^{1/2}\,(2\pi)^{-3/2}\int_{C_+} \exp\,\{i(x\cdot p + yq_- - t\omega_+)\} \times \times\;\overline{T_+(y,|p|,\lambda)}\;\ell(p,q)\,dpdq$$

where $q_- = q_-(|p|,\lambda)$, $\omega_+ = \omega_+(p,q)$ and $\lambda = \lambda(p,q) = \omega_+^2(p,q)$. This representation suggests that

$$v_\ell(t,\cdot) \sim v_\ell^2(t,\cdot) \text{ in } L_2(R^3_-(d)),\ t \to \infty \tag{3.21}$$

where

$$\begin{aligned} v_\ell^2(t,x,y) = c(\infty)\ \rho(\infty)^{1/2}\ (2\pi)^{-3/2} \int_{C_+} \exp\{i(x\cdot p + yq_- - t\omega_+)\} \times \\ \times\ \overline{T_+(|p|,\lambda)}\ \ell(p,q)dpdq \end{aligned} \tag{3.22}$$

Now the mapping $(p,q) \to (p,q') = X'(p,q) = (p,q_-(|p|,\omega_+^2(p,q)))$ with domain C_+ has range $X'(C_+) = R^3_+ = R^3_+(0)$, Jacobian $\partial(p,q)/\partial(p,q')$ $= c^2(-\infty)q'/c^2(\infty)q$ and satisfies $\omega_+(p,q) = \omega_-(p,q')$. Thus (3.22) implies the representation

$$\begin{aligned} v_\ell^2(t,x,y) = c(\infty)\ \rho(\infty)^{1/2}\ (2\pi)^{-3/2} \int_{R^3_+} \exp\{i(x\cdot p + yq' - t\omega_-)\} \times \\ \times\ \overline{T_+(|p|,\omega_-^2(p,q'))}\ \ell(p,q)(c^2(-\infty)q'/c^2(\infty)q)dpdq' \end{aligned} \tag{3.23}$$

where $q = q(|p|,q') = \sqrt{a^2(|p|^2 + q'^2) + q'^2}$. Note that

$$v_\ell^2(t,\cdot) = \exp(-i\,t\,c(-\infty)\ A_0^{1/2})\ h_\ell^2 \tag{3.24}$$

where $h_\ell^2 \in L_2(R^3)$ has Fourier transform

$$\hat{h}_\ell^2(p,q') = c(\infty)\ \rho(\infty)^{1/2}\ \overline{T_+(|p|,\omega_-^2(p,q'))}\ \ell(p,q)(c^2(-\infty)q'/c^2(\infty)q). \tag{3.25}$$

Since $\operatorname{supp} \hat{h}_\ell^2 \subset R^3_+$ the results of [9, Ch. 2] imply that $v_\ell^2(t,\cdot) \sim 0$ in $L_2(R^3_-(d))$ when $t \to \infty$. Combining this with (3.21) gives

$$v_\ell(t,\cdot) \sim 0 \text{ in } L_2(R^3_-(d)),\ t \to \infty. \tag{3.26}$$

Analogous conjectures concerning $v_m(t,\cdot)$ and $v_n(t,\cdot)$ will now be formulated. Only the main steps of the calculations will be given since the method is the same as for $v_\ell(t,\cdot)$.

<u>Behavior of v_m.</u> v_m has the representation

$$(3.27)\qquad v_m(t,x,y) = \int_{C_0} \phi_-(x,y,p,q)\ \exp\ (-i\,t\,\omega_+(p,q))\ m(p,q)\,dp\,dq,$$

by (3.5), where

$$(3.28)\qquad \phi_-(x,y,p,q) = (2\pi)^{-1} c(\infty)(2q)^{1/2}\, e^{ip\cdot x}\ \overline{\psi_0(y,|p|,\lambda)}$$

for $(p,q) = X_0(p,\lambda) \in C_0$ (and hence $\lambda = \lambda(p,q) = \omega_+^2(p,q)$). Moreover (see [12, (4.18)-(4.24)]),

$$(3.29)\qquad \psi_0(y,\mu,\lambda) = \left(\frac{\rho(\infty)}{4\pi q_+(\mu,\lambda)}\right)^{1/2} T_0(y,\mu,\lambda)\ \exp\ (y\,q_-'(\mu,\lambda))$$

where

$$(3.30)\qquad T_0(y,\mu,\lambda) = T_0(\mu,\lambda)\ \phi_3(y,\mu,\lambda)\exp\ (-y\,q_-'(\mu,\lambda)) \to T_0(\mu,\lambda),\ \ y \to -\infty$$

and

$$(3.31)\qquad \begin{aligned}\psi_0(y,\mu,\lambda) = \left(\frac{\rho(\infty)}{4\pi q_+(\mu,\lambda)}\right)^{1/2} [I_0(y,\mu,\lambda)\exp\{-i\,y\,q_+(\mu,\lambda)\}\\ + R_0(y,\mu,\lambda)\ \exp\ \{i\,y\,q_+(\mu,\lambda)\}]\end{aligned}$$

where

$$(3.32)\qquad \left.\begin{cases} I_0(y,\mu,\lambda) = \phi_2(y,\mu,\lambda)\exp\{i\,y\,q_+(\mu,\lambda)\} \to 1 \\ \\ R_0(y,\mu,\lambda) = R_0(\mu,\lambda)\ \phi_1(y,\mu,\lambda)\exp\{-i\,y\,q_+(\mu,\lambda)\} \to R_0(\mu,\lambda)\end{cases}\right\}\ y \to \infty.$$

Combining (3.27), (3.28) and (3.29) gives

$$v_m(t,x,y) = c(\infty)\ \rho(\infty)^{1/2}\ (2\pi)^{-3/2} \int_{C_0} \exp\ \{i(x\cdot p - t\omega_+)\} \times \tag{3.33}$$
$$\times\ \overline{T_0(y,|p|,\lambda)}\ \exp\ (y\, q_-')\ m(p,q)dpdq$$

where $\omega_+ = \omega_+(p,q)$, $\lambda = \omega_+^2(p,q)$ and $q_-' = q_-'(|p|,\lambda)$. Since

$$T_0(y,|p|,\lambda)\ \exp\ (y\ q_-'(|p|,\lambda)) \to 0,\ y \to -\infty, \tag{3.34}$$

equation (3.33) suggests that

$$v_m(t,\cdot) \sim 0 \text{ in } L_2(R_-^3(d)),\ t \to \infty. \tag{3.35}$$

Similarly, combining (3.27), (3.28) and (3.31) gives

$$\begin{aligned} v_m(t,x,y) = {} & c(\infty)\ \rho(\infty)^{1/2}\ (2\pi)^{-3/2} \int_{C_0} \exp\ \{i(x\cdot p + y\,q - t\omega_+)\} \times \\ & \times\ \overline{I_0(y,|p|,\lambda)}\ m(p,q)dpdq \\ & + c(\infty)\ \rho(\infty)^{1/2}\ (2\pi)^{-3/2} \int_{C_0} \exp\ \{i(x\cdot p - y\,q - t\omega_+)\} \times \\ & \times\ \overline{R_0(y,|p|,\lambda)}\ m(p,q)dpdq \end{aligned} \tag{3.36}$$

which suggests that

$$v_m(t,\cdot) \sim v_m^0(t,\cdot) + v_m^1(t,\cdot) \text{ in } L_2(R_+^3(d)),\ t \to \infty \tag{3.37}$$

where v_m^0 and v_m^1 are defined by

$$\begin{cases} v_m^0(t,\cdot) = \exp\ (-i\,t\,c(\infty)\ A_0^{1/2})\ h_m \\ v_m^1(t,\cdot) = \exp\ (-i\,t\,c(\infty)\ A_0^{1/2})\ h_m^1 \end{cases} \tag{3.38}$$

and h_m and h_m^1 are the functions in $L_2(R^3)$ whose Fourier transforms are

$$(3.39)\quad \begin{cases} \hat{h}_m(p,q) = c(\infty)\ \rho(\infty)^{1/2}\, m(p,q) = c(\infty)\ \rho(\infty)^{1/2}\, \chi_0(p,q)\ \hat{h}\_(p,q), \\ \\ \hat{h}_m^1(p,q) = c(\infty)\ \rho(\infty)^{1/2}\ \overline{R_0(|p|,\lambda)}\ m(p,-q) \end{cases}$$

$$= c(\infty)\ \rho(\infty)^{1/2}\ \overline{R_0(|p|,\lambda)}\ (1 - \chi_0(p,q))\ \hat{h}\_(p,-q)\ .$$

Note that supp $\hat{h}_m^1 \subset R_-^3$ and hence $v_m^1(t,\cdot) \sim 0$ in $L_2(R_+^3(d))$ when $t \to \infty$. Combining this with (3.37) gives

$$(3.40)\qquad v_m(t,\cdot) \sim v_m^0(t,\cdot) \text{ in } L_2(R_+^3(d)),\ t \to \infty.$$

<u>Behavior of v_n</u>. v_n has the representation

$$(3.41)\qquad v_n(t,x,y) = \int_{C_-} \phi\_(x,y,p,q)\ \exp\ (-i\,t\,\omega\_(p,q))\ n(p,q)dpdq,$$

by (3.5), where

$$(3.42)\qquad \phi\_(x,y,p,q) = (2\pi)^{-1} c(-\infty)(2|q|)^{1/2}\, e^{ip\cdot x}\ \overline{\psi\_(y,|p|,\lambda)}$$

for $(p,q) = X\_(p,\lambda) \in C\_$ (and hence $\lambda = \lambda(p,q) = \omega_-^2(p,q)$. Moreover (see [12, (4.5) and (4.12)-(4.17)]),

$$(3.43)\qquad \psi\_(y,\mu,\lambda) = \left(\frac{\rho(-\infty)}{4\pi q\_(\mu,\lambda)}\right)^{1/2} T\_(y,\mu,\lambda)\ \exp\ \{i\,y\,q_+(\mu,\lambda)\}$$

where

$$(3.44)\qquad T\_(y,\mu,\lambda) = T\_(\mu,\lambda)\ \phi_1(y,\mu,\lambda)\ \exp\{-i\,y\,q_+(\mu,\lambda)\} \to T\_(\mu,\lambda),\ \ y \to \infty$$

and

$$\psi_-(y,\mu,\lambda) = \left(\frac{\rho(-\infty)}{4\pi q_-(\mu,\lambda)}\right)^{1/2} [I_-(y,\mu,\lambda) \exp\{i y q_-(\mu,\lambda)\}$$

$$(3.45) \qquad + R_-(y,\mu,\lambda) \exp\{-i y q_-(\mu,\lambda)\}]$$

where

$$(3.46) \quad \left.\begin{cases} I_-(y,\mu,\lambda) = \phi_3(y,\mu,\lambda) \exp\{-i y q_-(\mu,\lambda)\} \to 1 \\ \\ R_-(y,\mu,\lambda) = R_-(\mu,\lambda)\ \phi_4(y,\mu,\lambda) \exp\{i y q_-(\mu,\lambda)\} \to R_-(\mu,\lambda) \end{cases}\right\} y \to -\infty.$$

Combining (3.41), (3.42) and (3.45) gives, after simplification using $q_-(|p|,\omega_-^2(p,q)) = (q^2)^{1/2} = -q$ for $(p,q) \in C_-$,

$$v_n(t,x,y) = c(-\infty)\ \rho(-\infty)^{1/2}\ (2\pi)^{-3/2} \int_{C_-} \exp\{i(x\cdot p + y q - t\omega_-)\} \times$$

$$\times\ \overline{I_-(y,|p|,\lambda)}\ n(p,q)dpdq$$

$$(3.47) \qquad + c(-\infty)\ \rho(-\infty)^{1/2}\ (2\pi)^{-3/2} \int_{C_-} \exp\{i(x\cdot p - y q - t\omega_-)\}$$

$$\times\ \overline{R_-(y,|p|,\lambda)}\ n(p,q)dpdq.$$

This suggests the asymptotic behavior

$$(3.48) \qquad v_n(t,\cdot) \sim v_n^0(t,\cdot) + v_n^1(t,\cdot) \text{ in } L_2(R_-^3(d)),\ t \to \infty,$$

where

$$(3.49) \qquad \begin{cases} v_n^0(t,\cdot) = \exp(-i t c(-\infty) A_0^{1/2})\ h_n \\ \\ v_n^1(t,\cdot) = \exp(-i t c(-\infty) A_0^{1/2})\ h_n^1 \end{cases}$$

and h_n and h_n^1 are the functions whose Fourier transforms are

$$(3.50)\quad \begin{cases} \hat{h}_n(p,q) = c(-\infty)\ \rho(-\infty)^{1/2}\, n(p,q) = c(-\infty)\ \rho(-\infty)^{1/2}\, \chi_-(p,q)\ \hat{h}_-(p,q), \\ \hat{h}_n^1(p,q) = c(-\infty)\ \rho(-\infty)^{1/2}\, \overline{R_-(|p|,\lambda)}\ n(p,-q) \\ \qquad = c(-\infty)\ \rho(-\infty)^{1/2}\, \overline{R_-(|p|,\lambda)}\ (1 - \chi_-(p,q))\ \hat{h}_-(p,-q). \end{cases}$$

Note that supp $\hat{h}_n^1 \subset R_+^3$ and hence $v_n^1(t,\cdot) \sim 0$ in $L_2(R_-^3(d))$ when $t \to \infty$. Combining this with (3.48) gives

$$(3.51)\qquad v_n(t,\cdot) \sim v_n^0(t,\cdot) \text{ in } L_2(R_-^3(d)),\ t \to \infty.$$

Finally, combining (3.41), (3.42) and (3.43) gives

$$(3.52)\quad \begin{aligned} v_n(t,x,y) = c(-\infty)\ \rho(-\infty)^{1/2}\ (2\pi)^{-3/2} \int_{C_-} \exp\{i(x\cdot p - y\,q_+ - t\omega_-)\} \times \\ \times\ \overline{T_-(y,|p|,\lambda)}\ n(p,q)\,dp\,dq \end{aligned}$$

where $\omega_- = \omega_-(p,q)$, $\lambda = \omega_-^2(p,q)$ and $q_+ = q_+(|p|,\lambda)$. This suggests that

$$(3.53)\qquad v_n(t,\cdot) \sim v_n^2(t,\cdot) \text{ in } L_2(R_+^3(d)),\ t \to \infty$$

where

$$(3.54)\quad \begin{aligned} v_n^2(t,x,y) = c(-\infty)\ \rho(-\infty)^{1/2}\ (2\pi)^{-3/2} \int_{C_-} \exp\{i(x\cdot p - y\,q_+ - t\omega_-)\} \\ \times\ \overline{T_-(|p|,\lambda)}\ n(p,q)\,dp\,dq. \end{aligned}$$

Now the mapping $(p,q) \to (p,q') = X''(p,q) = (p,-q_+(p,\lambda(p,q)))$ maps C_- onto $X''(C_-) = -C_+$, has Jacobian $\partial(p,q)/\partial(p,q') = c^2(\infty)q'/c^2(-\infty)q$ and satisfies $\omega_-(p,q) = \omega_+(p,q')$. Thus

$$v_n^2(t,\cdot) = \exp\,(-i\,t\,c(\infty)\,A_0^{1/2})\,h_n^2 \tag{3.55}$$

where h_n^2 has the Fourier transform

$$\hat{h}_n^2(p,q') = c(-\infty)\,\rho(-\infty)^{1/2}\,\overline{T_-(|p|,\omega_+^2(p,q'))}\,n(p,q(p,q')) \times \times\,(c^2(\infty)q'/c^2(-\infty)q). \tag{3.56}$$

Moreover, supp $\hat{h}_n^2 \subset R_-^3$ and hence $v_n^2(t,\cdot) \sim 0$ in $L_2(R_+^3(d))$, $t \to \infty$. Combining this with (3.53) gives

$$v_n(t,\cdot) \sim 0 \text{ in } L_2(R_+^3(d)),\ t \to \infty. \tag{3.57}$$

The asymptotic behavior of $v_f(t,\cdot)$ for $t \to \infty$ may be obtained from the three cases analyzed above by superposition, equation (3.4). Thus equations (3.19), (3.26), (3.35), (3.40), (3.51) and (3.57) imply

$$v_f(t,\cdot) \sim \left\{\begin{array}{ll} v_\ell^0(t,\cdot) + v_m^0(t,\cdot) & \text{in } L_2(R_+^3(d)) \\ v_n^0(t,\cdot) & \text{in } L_2(R_-^3(d)) \end{array}\right\}\ t \to \infty. \tag{3.58}$$

On combining this with the definitions of v_ℓ^0, v_m^0 and v_n^0, equations (3.17), (3.18), (3.38), (3.39), (3.49) and (3.50), one is led to formulate

<u>Theorem 3.1</u>. For every $h \in \mathcal{H}$ let $v_f^0(t,\cdot)$ be defined by

$$v_f^0(t,x,y) = \begin{cases} \exp\,(-i\,t\,c(\infty)\,A_0^{1/2})\,h^+(x,y), & (x,y) \in R_+^3(d), \\ \exp\,(-i\,t\,c(-\infty)\,A_0^{1/2})\,h^-(x,y), & (x,y) \in R_-^3(d), \end{cases} \tag{3.59}$$

where h^+ and h^- are the functions in $L_2(R^3)$ whose Fourier transforms are given by

$$\hat{h}^{+}(p,q) = \begin{cases} c(\infty)\ \rho(\infty)^{1/2}\ \hat{h}_{-}(p,q), & (p,q) \in R^3_+, \\ 0 & , (p,q) \in R^3_-, \end{cases} \tag{3.60}$$

and

$$\hat{h}^{-}(p,q) = \begin{cases} 0 & , (p,q) \in R^3_+, \\ c(-\infty)\ \rho(-\infty)^{1/2}\ \hat{h}_{-}(p,q), & (p,q) \in R^3_-. \end{cases} \tag{3.61}$$

Then

$$\lim_{t\to\infty} \| v_f(t,\cdot) - v_f^0(t,\cdot)\|_{\mathcal{H}} = 0. \tag{3.62}$$

Theorem 3.1 implies corresponding asymptotic estimates for the free component $u_f(t,\cdot) = P_f u(t,\cdot) = \mathrm{Re}\ \{v_f(t,\cdot)\}$ of the acoustic potential $u(t,\cdot)$. Indeed, if $u_f^0(t,\cdot)$ is defined by

$$u_f^0(t,\cdot) = \mathrm{Re}\ \{v_f^0(t,\cdot)\} \tag{3.63}$$

then Theorem 3.1 and the elementary inequality $|\mathrm{Re}\ z| \le |z|$ imply

<u>Corollary 3.2</u>. For all $h \in \mathcal{H}$ one has

$$\lim_{t\to\infty} \| u_f(t,\cdot) - u_f^0(t,\cdot)\|_{\mathcal{H}} = 0. \tag{3.64}$$

If the initial state h has derivatives in $\mathcal{H}$ then $u_f(t,\cdot)$ and $u_f^0(t,\cdot)$ have the same derivatives in $\mathcal{H}$ and (3.64) can be strengthened to include these derivatives. In particular, one has

<u>Corollary 3.3</u>. For all $h \in L_2^1(R^3) = D(A^{1/2})$ one has

$$\lim_{t\to\infty} \| D_j\ u_f(t,\cdot) - D_j\ u_f^0(t,\cdot)\|_{\mathcal{H}} = 0,\ j = 0,1,2,3, \tag{3.65}$$

where $D_0 = \partial/\partial t$, $D_1 = \partial/\partial x_1$, $D_2 = \partial/\partial x_2$ and $D_3 = \partial/\partial y$. Equation (3.65) is equivalent to convergence in energy:

$$\lim_{t\to\infty} E(u_f - u_f^0, R^3, t) = 0. \tag{3.66}$$

Corollary 3.3 can be proved by applying the method of this section to the derivatives $D_j\, v_f(t,\cdot)$ $(j = 0,1,2,3)$ which are given by integrals of the same form as (3.2). Detailed proofs for the case of the Pekeris profile were given in [10].

Proof of Theorem 3.1. The remainder of this section is devoted to the proof of Theorem 3.1. The decomposition (3.3) is used for the proof. Moreover, for brevity, only the asymptotic equality (3.53) for $v_n(t,\cdot)$ is proved. The remaining five cases, namely (3.14), (3.21), (3.35), (3.40) and (3.48) can be proved by the method used for (3.53). As a first step, (3.53) will be proved for the special case of $n(p,q) \in C_0(C_-)$, the set of continuous function with compact supports in the open cone C_-. The general case will then be proved by using the fact that $C_0(C_-)$ is dense in $L_2(C_-)$.

For functions $n(p,q) \in C_0(C_-)$ the integrals defining v_n and v_n^2 converge point-wise, as well as in $\mathcal{H}$, and one can write

$$\begin{aligned} &v_n(t,x,y) - v_n^2(t,x,y) \\ &\quad = c(-\infty)\, \rho(-\infty)^{1/2}\, (2\pi)^{-3/2} \int_{R^2} \exp\,(ix\cdot p)\; w(y,p,t)\,dp \end{aligned} \tag{3.67}$$

where

$$(3.68)\quad w(y,p,t) = \int_{-\infty}^{0} \exp\{-i(y q_+ + t\omega_-)\}\, [\overline{T_-(y,|p|,\omega_-^2)} - \overline{T_-(|p|,\omega_-^2)}]\, n(p,q)\, dq.$$

Parseval's formula in $L_2(R^2)$, applied to (3.67), gives

$$(3.69)\quad \int_{R^2} |v_n(t,x,y) - v_n^2(t,x,y)|^2 dx = c^2(-\infty)\, \rho(-\infty)\,(2\pi)^{-1} \int_{R^2} |w(y,p,t)|^2\, dp.$$

On integrating this over $y \geq d$ one finds

$$(3.70)\quad \| v_n(t,\cdot) - v_n^2(t,\cdot) \|_{L_2(R_+^3(d))} = c(-\infty)\, \rho(-\infty)^{1/2} (2\pi)^{-1/2} \| w(\cdot,t) \|_{L_2(R_+^3(d))}.$$

The last relation implies that to prove (3.53) it is sufficient to prove that

$$(3.71)\qquad w(\cdot,t) \to 0 \text{ in } L_2(R_+^3(d)),\ t \to \infty.$$

To this end it will be convenient to change the variable of integration in (3.68) from q to $\omega = \omega_-(p,q) = c(-\infty)\sqrt{|p|^2 + q^2}$. Solving this equation for $q < 0$ gives $q = -(\omega^2 c^{-2}(-\infty) - |p|^2)^{1/2} = -q_-(|p|,\omega^2)$ with $\omega > c(-\infty)|p|$. Hence (3.68) can be written

$$(3.72)\qquad w(y,p,t) = \int_{c(-\infty)|p|}^{\infty} \exp(-i t \omega)\, W(y,p,\omega)\, d\omega$$

where

$W(y,p,\omega)$

$$(3.73)\quad = \frac{c^{-2}(-\infty) \exp\{-i y q_+(|p|,\omega^2)\} [\overline{T_-(y,|p|,\omega^2)} - \overline{T_-(|p|,\omega^2)}]\, n(p,-q_-(|p|,\omega^2))\,\omega}{q_-(|p|,\omega^2)}.$$

The assumption that $n \in C_0(C_-)$, together with (3.44), implies that $W \in C_0(R \times \Gamma)$ where $\Gamma = \{(p,\omega) : \omega > c(-\infty)|p|\}$. Moreover, by a standard partition of unity argument, one may assume without loss of generality that

$$\text{supp } W(y,\cdot) \subset \{(p,\omega) : |p| \leq p_0 \text{ and } 0 < \omega_0 \leq \omega \leq \omega_1\} \tag{3.74}$$

for all $y \in R$ where $\omega_0 > c(-\infty)p_0$. This in turn implies that

$$w(y,p,t) = \int_{\omega_0}^{\omega_1} \exp(-i t \omega) \, W(y,p,\omega) d\omega \tag{3.75}$$

and

$$\text{supp } w(y,\cdot,t) \subset B(p_0) = \{p : |p| \leq p_0\} \tag{3.76}$$

for all $y \in R$ and $t \in R$. Thus

$$\begin{aligned} \|w(\cdot,t)\|^2_{L_2(R^3_+(d))} &= \int_d^\infty \int_{B(p_0)} |w(y,p,t)|^2 \, dpdy \\ &= \int_d^{y_0} \int_{B(p_0)} |w(y,p,t)|^2 \, dpdy \\ &\quad + \int_{y_0}^\infty \int_{B(p_0)} |w(y,p,t)|^2 \, dpdy \end{aligned} \tag{3.77}$$

for any $y_0 > d$. The proof of (3.71) will be derived from (3.77) and the following two lemmas.

<u>Lemma 3.4</u>. Let $n \in C_0(C_-)$ and assume that (3.74) holds. Then for each $d \in R$, $y_0 > d$ and $p_0 > 0$ one has

$$\lim_{t\to\infty} w(y,p,t) = 0, \tag{3.78}$$

uniformly for all $(y,p) \in [d,y_0] \times B(p_0)$.

Lemma 3.5. Under the hypotheses of Lemma 3.4 there is a constant $C = C(n)$ such that

$$|w(y,p,t)| \leq C\, y^{1-\alpha} \tag{3.79}$$

for all $y > 0$, $p \in B(p_0)$ and $t \in R$, where $\alpha > 3/2$ is the constant of condition (1.6).

Proof of Lemma 3.4. The proof is based on a well-known proof of the Riemann-Lebesgue lemma. Note that by (3.75) one has

$$\begin{aligned} w(y,p,t) &= -\int_{\omega_0-(\pi/t)}^{\omega_1-(\pi/t)} \exp\,(-i\omega\, t)\; W(y,p,\omega+(\pi/t))\,d\omega \\ &= \frac{1}{2}\int_{\omega_0}^{\omega_1} \exp\,(-i\,t\,\omega)\,[W(y,p,\omega) - W(y,p,\omega+(\pi/t))]\,d\omega \\ &\quad - \frac{1}{2}\int_{\omega_0-(\pi/t)}^{\omega_0} \exp\,(-i\,t\,\omega)\; W(y,p,\omega+(\pi/t))\,d\omega \\ &\quad + \frac{1}{2}\int_{\omega_1-(\pi/t)}^{\omega_1} \exp\,(-i\,t\,\omega)\; W(y,p,\omega+(\pi/t))\,d\omega. \end{aligned} \tag{3.80}$$

The limit relation (3.78) is obvious from (3.80) and the continuity of W. The uniformity of the limit follows from (3.80) and the uniform continuity of W on compact subsets of $R \times \Gamma$.

Proof of Lemma 3.5. Note that by (3.72), (3.73) and (3.44) one has the estimate

$$(3.81) \qquad |w(y,p,t)| \le \int_{\omega_0}^{\omega_1} |\phi_1(y,|p|,\omega^2)\exp\{-iy\,q_+(|p|,\omega^2)\} - 1|\; M(p,\omega)\,d\omega$$

for all y, p and t where

$$(3.82) \qquad M(p,\omega) = c^{-2}(-\infty)\;|T_-(|p|,\omega^2)|\;\left|\frac{n(p,-q_-(|p|,\omega^2))\omega}{q_-(|p|,\omega^2)}\right|$$

is continuous for $|p| \le p_0$, $\omega_0 \le \omega \le \omega_1$. It follows that for any $y \in R$, $p \in B(p_0)$ and $t \in R$

$$(3.83) \qquad |w(y,p,t)| \le M_0 \sup |\phi_1(y,|p|,\omega^2)\exp\{-iy\,q_+(|p|,\omega^2)\} - 1|$$

where $M_0 = M_0(n) = \sup M(p,\omega)$ and the suprema are taken over all $|p| \le p_0$ and $\omega_0 \le \omega \le \omega_1$. The proof of (3.79) will be based on (3.83), conditions (1.3), (1.4), (1.6) on $\rho(y)$ and $c(y)$ and the proof in [12] of Theorem 2.1; see [12, p. 27ff]. Note that because of the continuity of w it will suffice to prove (3.79) for all $y > y_1$ where $y_1 = y_1(n)$ is a positive constant.

The solution $\phi_1(y,\mu,\lambda)$ with $\lambda > c^2(-\infty)\mu^2 \ge c^2(\infty)\mu^2$ satisfies [12, p. 27ff]

$$(3.84) \qquad \phi_1(y,\mu,\lambda) = \exp\{iy\,q_+(\mu,\lambda)\}\,(\eta_1 + \eta_2)$$

where $\eta = (\eta_1,\eta_2)$ is characterized on $y \ge y_1$ as the unique solution of the integral equation [12, (2.64)] which can be written

$$(3.85) \qquad \eta = \eta^0 + K(\mu,\lambda)\eta,\quad \eta^0 = (1,0).$$

The kernel $K(y,y',\mu,\lambda) = (K_{ij}(y,y',\mu,\lambda))$ is defined by

$$(3.86)\qquad K_{1j}(y,y',\mu,\lambda) = \begin{cases} 0 & , \ y_1 \le y' < y, \\ -E_{1j}(y',\mu,\lambda), & y' \ge y, \end{cases}$$

and

$$(3.87)\qquad K_{2j}(y,y',\mu,\lambda) = \begin{cases} 0 & , \ y_1 \le y' < y, \\ -\exp\{-2i(y-y')q_+(\mu,\lambda)\ E_{2j}(y',\mu,\lambda), & y' \ge y, \end{cases}$$

where [12, p. 27]

$$(3.88)\qquad E(y,\mu,\lambda) = B^{-1}(\mu,\lambda)\ N(y,\mu,\lambda)\ B(\mu,\lambda).$$

From these relations one has [12, (2.64)]

$$\phi_1(y,\mu,\lambda)\exp\{-i\,y\,q_+(\mu,\lambda)\} - 1 = \eta_1 + \eta_2 - 1$$

(3.89)

$$= -\int_y^\infty E_{1j}(y',\mu,\lambda)\ \eta_j(y')dy' - \int_y^\infty \exp\{-2i(y-y')q_+(\mu,\lambda)\}\ E_{2j}(y',\mu,\lambda)\eta_j(y')dy'$$

and hence

$$(3.90)\qquad |\phi_1(y,\mu,\lambda)\exp\{-i\,y\,q_+(\mu,\lambda)\} - 1| \le \sum_{j,k=1}^{2} \int_y^\infty |E_{jk}(y',\mu,\lambda)|\,|\eta_k(y')|dy'.$$

Using [12, (2.66)] and the continuity of $B(\mu,\lambda)$ on $\lambda > c^2(-\infty)\mu^2$ it can be shown that

$$(3.91)\qquad \|K(\mu,\lambda)\| \le 1/2 \text{ for } 0 \le \mu \le p_0,\ \omega_0^2 \le \lambda \le \omega_1^2$$

provided $y_1 = y_1(n)$ is large enough. Thus

$$\|\eta\| \le \frac{\eta^0}{1 - \|K\|} = \frac{1}{1 - \|K\|} \le 2 \tag{3.92}$$

for $0 \le \mu \le p_0$, $\omega_0^2 \le \lambda \le \omega_1^2$. Combining this with (3.90) and [12, (2.66)] gives

$$\begin{aligned} |\phi_1(y,\mu,\lambda) \exp\{-i\, y\, q_+(\mu,\lambda)\} - 1| &\le 2 \int_y^\infty \sum_{j,k=1}^{2} |E_{jk}(y',\mu,\lambda)|\, dy' \\ &\le C_1 \int_y^\infty |\rho(y') - \rho(\infty)|\, dy' \\ &\quad + C_2 \int_y^\infty |c(y') - c(\infty)|\, dy' \end{aligned} \tag{3.93}$$

for the same values of μ and λ where C_1 and C_2 depends only on p_0, ω_0 and ω_1. It follows from (3.93) and (1.6) that

$$\sup |\phi_1(y,|p|,\omega^2) \exp\{-i\, y\, q_+(|p|,\omega^2)\} - 1| \le C_3\, y^{1-\alpha} \tag{3.94}$$

for all $y \ge y_1(n)$ where C_3 depends only on p_0, ω_0 and ω_1 (i.e., n). Combining (3.94) with (3.81) gives (3.75).

Proof of Theorem 3.1 (completed). Lemma 3.5 implies that

$$\int_{y_0}^\infty \int_{B(p_0)} |w(y,p,t)|^2\, dp dy \le (\pi p_0^2\, C^2/2\alpha - 3)\, y_0^{3-2\alpha} \tag{3.95}$$

where $3 - 2\alpha < 0$. Thus given any $\varepsilon > 0$ there is a $y_0 = y_0(\varepsilon,n)$ such that

$$\int_{y_0}^\infty \int_{B(p_0)} |w(y,p,t)|^2\, dp dy \le \varepsilon \tag{3.96}$$

for all $t \in R$. Equation (3.77), together with Lemma 3.4 and the estimate (3.96), imply that

$$\text{(3.97)} \qquad \limsup_{t\to\infty} \|w(\cdot,t)\|^2_{L_2(R^3_+(d))} \le \varepsilon .$$

Since $\varepsilon > 0$ is arbitrary this implies (3.71), as required.

The arguments given above, applied to v_ℓ, v_m and v_n, show that the conclusions of Theorem 3.1 hold for all h such that $\hat{h}_- \in C_0(C_+ \cup C_0 \cup C_-)$. Moreover, this set is dense in $L_2(R^3)$ and hence $\Phi^*_- C_0(C_+ \cup C_0 \cup C_-)$ is dense in $\mathcal{H}_f = P_f\,\mathcal{H}$ by [12, Cor. 8.12]. These facts can be used to extend (3.62) to all $h \in \mathcal{H}$ because the mappings $U(t) : \mathcal{H}_f \to L_2(R^3)$ and $U_0(t) : \mathcal{H}_f \to L_2(R^3)$ defined by $U(t)h_f = \exp(-i\,t\,A^{1/2})\,h_f$ and $U_0(t)h_f = v^0_f(t,\cdot)$ are uniformly bounded for all $t \in R$ (see [11, p. 32]). The density argument needed to extend (3.62) to all $h \in \mathcal{H}$ has been given in many places; see, for example, [9, Ch. 2] or [10, p. 260].

BLANK PAGE

§4. Transient Guided Waves.

The asymptotic behavior for $t \to \infty$ of the guided component $u_g(t,\cdot) = P_g\, u(t,\cdot)$ is derived in this section. $u_g(t,\cdot)$ is a sum in $\mathcal{H}$ of mutually orthogonal partial waves $u_k(t,\cdot) = P_k\, u(t,\cdot) = \mathrm{Re}\,\{v_k(t,\cdot)\}$, $1 \le k < N_0$. The starting point for the analysis is the integral representation

$$(4.1) \qquad v_k(t,x,y) = \int_{\Omega_k} \psi_k(x,y,p) \exp\{-i\,t\,\omega_k(|p|)\}\, \tilde{h}_k(p)\,dp$$

where

$$(4.2) \qquad \tilde{h}_k(p) = \int_{R^3} \overline{\psi_k(x,y,p)}\, h(x,y)\, c^{-2}(y)\, \rho^{-1}(y)\,dx\,dy$$

and the integrals converge in $\mathcal{H}$ and $L_2(\Omega_k)$, respectively. The integral in (4.1) can be written

$$(4.3) \qquad v_k(t,x,y) = \frac{1}{2\pi} \int_{\Omega_k} \exp\{i(x\cdot p - t\,\omega_k(|p|))\}\, \psi_k(y,p)\, \tilde{h}_k(p)\,dp.$$

This is an oscillatory integral that can be estimated by the method of stationary phase when $\tilde{h}_k \in C_0^\infty(\Omega_k)$. To apply the method define

$$(4.4) \qquad \begin{cases} r = \sqrt{t^2 + |x|^2} \\ t = r\,\xi_0,\ x_1 = r\,\xi_1,\ x_2 = r\,\xi_2 \\ \xi = (\xi_0,\xi_1,\xi_2) \in S^2 \subset R^3 \end{cases}$$

where S^2 denotes the unit sphere in R^3. Then (4.3) takes the form

$$v_k(t,x,y) = \int_{\Omega_k} \exp\{i\,r\,\theta_k(p,\xi)\}\, g_k(p,y)\, dp \tag{4.5}$$

where

$$\begin{cases} \theta_k(p,\xi) = \xi_1 p_1 + \xi_2 p_2 - \xi_0\, \omega_k(|p|) \\ \\ g_k(p,y) = (2\pi)^{-1}\, \psi_k(y,p)\, \tilde{h}_k(p) \end{cases} \tag{4.6}$$

Estimates for large r of the integral in (4.5) are needed that are uniform for (ξ,y) in compact subsets of $S^2 \times R$. Such estimates are provided by a version of the method of stationary phase due to M. Matsumura [5]. A form of Matsumura's results applicable to (4.5) was presented in [11, Appendix]. This result is applied below to estimating v_k.

The phase function θ_k has a point of stationary phase if and only if

$$U_k(|p|)p/|p| = x/t \tag{4.7}$$

where

$$U_k(|p|) = \omega_k'(|p|) \tag{4.8}$$

is the group speed associated with the dispersion relation $\omega = \omega_k(|p|)$. It will be assumed, for brevity, that $U_k(\mu)$ is a monotone decreasing function that maps $\mathcal{O}_k$ onto $(c_m, c(\infty))$. In this case (4.7) has a unique solution if $|x|/t$ lies in the range of $U_k(|p|)$; that is

$$c_m < |x|/t < c(\infty), \tag{4.9}$$

and no solution otherwise. The solution is given by

$$p = Q_k(|x|/t)\; x/|x| \tag{4.10}$$

where Q_k is the inverse function to U_k. By calculating the Hessian θ_k'' one can show that

$$r^2\; |\det \theta_k''(p,\xi)| = t^2\; U_k(|p|)\; |U_k'(|p|)|/|p| \tag{4.11}$$

and sgn $\theta_k''(p,\xi) = 0$. In particular, each point of stationary phase is non-degenerate and makes a contribution

$$v_k^\infty(t,x,y,p) = \frac{|p|^{1/2} \exp\{i(|x|\,|p|-t\omega_k(|p|))\}\; \psi_k(y,|p|)\tilde{h}_k(p)}{t\{U_k(|p|)\,|U_k'(|p|)|\}^{1/2}} \tag{4.12}$$

to the integral in (4.3), where p is given by (4.10). For $|x|/t$ outside the interval (4.9) there is no point of stationary phase. Thus the stationary phase approximation to $v_k(t,x,y)$ is given by

$$v_k^\infty(t,x,y) = \chi(|x|/t)\; v_k^\infty(t,x,y,Q_k(|x|/t)\; x/|x|) \tag{4.13}$$

where χ is the characteristic function of the interval $(c_m, c(\infty))$ and one has

<u>Theorem 4.1</u>. For all $h \in \mathcal{H}$ such that $\tilde{h}_k \in C_0^\infty(\Omega_k)$ there exists a constant $C = C_k(h)$ such that

$$|v_k(t,x,y) - v_k^\infty(t,x,y)| \le C/t^2 \tag{4.14}$$

for all $t > 0$, $x \in R^2 - \{0\}$ and $y \in R$.

Theorem 4.1 can be proved by application of Theorems A.1 and A.2 of [11]. The proof that $\psi_k(y,p)$ has the required p-derivatives is lengthy but straightforward and will not be given here. If $\tilde{h}_k$ is not a smooth function then Theorems A.1 and A.2 are not applicable and the estimate (4.14) may fail. However, the definitions (4.12) and (4.13) are meaningful for all $h \in \mathcal{H}$ and one has

<u>Theorem 4.2</u>. For all $h \in \mathcal{H}$, all $t > 0$ and $k = 1, 2, 3, \cdots$ one has

$$v_k^\infty(t,\cdot) \in \mathcal{H} \tag{4.15}$$

and

$$\| v_k^\infty(t,\cdot)\|_{\mathcal{H}} = \|\tilde{h}_k\|_{L_2(\Omega_k)} = \|P_k h\|_{\mathcal{H}} . \tag{4.16}$$

Moreover, the mapping $t \to v_k^\infty(t,\cdot)$ is continuous from R_+ to $\mathcal{H}$ and

$$\lim_{t\to\infty} \| v_k(t,\cdot) - v_k^\infty(t,\cdot)\|_{\mathcal{H}} = 0. \tag{4.17}$$

The proofs of these properties are the same as those for the Pekeris profile, given in [10], and are not reproduced here. On defining

$$u_k^\infty(t,x,y) = \mathrm{Re}\,\{v_k^\infty(t,x,y)\} \tag{4.18}$$

one also has

<u>Corollary 4.3</u>. For all $h \in \mathcal{H}$ and $k = 1, 2, 3, \cdots$,

$$\lim_{t\to\infty} \| u_k(t,\cdot) - u_k^\infty(t,\cdot)\|_{\mathcal{H}} = 0. \tag{4.19}$$

If $h \in L_2^1(R^3)$ then $u_k(t,\cdot) \in L_2^1(R^3)$ and asymptotic wave functions for the first derivatives of u_k can be constructed. Indeed, if

$\tilde{h}_k \in C_0^\infty(\Omega_k)$ then the first derivatives of v_k are given by

$$D_t\ v_k(t,x,y) = \frac{1}{2\pi}\int_{\Omega_k} \exp\{i(x\cdot p - t\ \omega_k(|p|))\}(-i\omega_k(|p|))\psi_k(y,p)\tilde{h}_k(p)dp,$$

(4.20)

$$D_j\ v_k(t,x,y) = \frac{1}{2\pi}\int_{\Omega_k} \exp\{i(x\cdot p - t\ \omega_k(|p|))\ (ip_j)\ \psi_k(y,p)\ \tilde{h}_k(p)dp\ (j = 1,2),$$

$$D_y\ v_k(t,x,y) = \frac{1}{2\pi}\int_{\Omega_k} \exp\{i(x\cdot p - t\ \omega_k(|p|))\}\ D_y\ \psi_k(y,p)\ \tilde{h}_k(p)dp.$$

These integrals have the same form as the integral (4.3) for v_k. The corresponding asymptotic wave functions are defined by

$$v_{k0}^\infty(t,x,y,p) = (-i\ \omega_k(|p|))\ v_k^\infty(t,x,y,p),$$

(4.21)

$$v_{kj}^\infty(t,x,y,p) = (ip_j)\ v_k^\infty(t,x,y,p)\ (j = 1,2),$$

$$v_{k3}^\infty(t,x,y,p) = D_y\ v_k^\infty(t,x,y,p),\ \text{and}$$

$$v_{kj}^\infty(t,x,y) = \chi(|x|/t)\ v_{kj}^\infty(t,x,y,Q_k(|x|/t)\ x/|x|)$$

for $j = 0,1,2,3$. The analogue of Theorem 4.2 is

<u>Theorem 4.4</u>. For all $h \in L_2^1(R^3)$, all $t > 0$ and $k = 1,2,3,\cdots$ one has

(4.22) $$v_{kj}^\infty(t,\cdot) \in L_2(R^3),\ j = 0,1,2,3,$$

(4.23) $$\|v_{k0}^\infty(t,\cdot)\|_{\mathcal{H}}^2 + \sum_{j=1}^{3} \|v_{kj}^\infty(t,\cdot)\|_{L_2(R^3,\rho^{-1}dxdy)}^2 = 2\ \|A^{1/2}\ h_k\|_{\mathcal{H}}^2,$$

and

$$(4.24) \qquad \lim_{t\to\infty} \| D_j \, v_k(t,\cdot) - v_{kj}^{\infty}(t,\cdot)\|_{L_2(R^3)} = 0, \; j = 0,1,2,3.$$

The proof of Theorem 4.4 is the same as for the special case of the Pekeris profile which was treated in detail in [10].

The preceding discussion was restricted to the special case where $U_k(\mu)$ is monotonic. If $U_k(\mu)$ has a finite number of maxima and minima there are a corresponding number of points of stationary phase and the form of the asymptotic wave function is more complicated but still tractable. In the case of the Pekeris profile, treated in [10], there are two points of stationary phase. Cases that lead to infinitely many stationary points have not yet been encountered. They would require additional analysis.

§5. Asymptotic Distributions of Energy for Large Times.

The total energy of the acoustic field $u(t,x,y)$, given by (2.14), is constant for $t \geq T$. The same is true of the partial waves u_f, u_g and u_k, $k = 1, 2, 3, \cdots$. Moreover, it was shown in [12] that $\{P_f, P_1, P_2, \cdots\}$ is a complete family of orthogonal projections in $\mathcal{H}$ that reduces A. It follows that

$$(5.1) \qquad \|A^{1/2} h\|_{\mathcal{H}}^2 = \|A^{1/2} h_f\|_{\mathcal{H}}^2 + \sum_{k=1}^{N_0-1} \|A^{1/2} h_k\|_{\mathcal{H}}^2$$

which may be interpreted as an energy partition theorem. The partial energies

$$(5.2) \qquad \begin{cases} E(u_f, R^3, t) = \|A^{1/2} h_f\|^2, \\ \\ E(u_k, R^3, t) = \|A^{1/2} h_k\|_{\mathcal{H}}^2, \; k = 1, 2, 3, \cdots, N_0-1, \end{cases}$$

can be calculated from the source function $f(t,x,y)$ and the normal mode functions. The relationship between h and f is given by (2.7), which implies that

$$(5.3) \qquad \hat{h}_-(p,q) = i\, \lambda^{-1/2}(p,q)\, \hat{f}_-(\lambda^{1/2}(p,q),p,q)$$

where

$$(5.4) \quad \hat{f}_-(\omega,p,q) = \int_{-\infty}^{\infty} \int_{R^3} \exp(i\omega\tau)\, \overline{\phi_-(x,y,p,q)}\, f(\tau,x,y)\, c^{-2}(y)\, \rho^{-1}(y) dydxd\tau$$

and similarly

$$\tilde{h}_k(p) = i\,\omega_k^{-1}(|p|)\,\tilde{f}_k(\omega_k(|p|),p) \tag{5.5}$$

where

$$\tilde{f}_k(\omega,p) = \int_{-\infty}^{\infty}\int_{R^3} \exp\,(i\,\omega\,\tau)\,\overline{\psi_k(x,y,p)}\,f(\tau,x,y)\,c^{-2}(y)\,\rho^{-1}(y)dydxd\tau. \tag{5.6}$$

It follows that the partial energies are given by

$$E(u_f,R^3,t) = \int_{R^3} |\hat{f}_-(\lambda^{1/2}(p,q),p,q)|^2\,dpdq \tag{5.7}$$

and

$$E(u_k,R^3,t) = \int_{\Omega_k} |\tilde{f}_k(\omega_k(|p|),p)|^2\,dp,\quad k = 1,\,2,\,3,\cdots, \tag{5.8}$$

for every $t \geq T$.

The theorems of §3 and §4 make it possible to calculate asymptotic distributions of energy in bounded and unbounded subsets of R^3. Only the principal results are formulated here. The proofs are omitted since they are the same as those for the Pekeris profile which were given in [10].

The notation

$$E^{\infty}(u,K) = \lim_{t\to\infty} E(u,K,t) \tag{5.9}$$

will be used whenever the limit exists. A first result is the transiency of all waves with finite energy in stratified fluids:

$$E^{\infty}(u,K) = 0 \text{ for all compact sets } K \subset R^3. \tag{5.10}$$

The Free Component. The results of §3 imply that in each half-space $R^3_+(d)$ (resp. $R^3_-(d)$) u_f behaves like a wave in a homogeneous medium. Thus if C^+ (resp. C^-) denotes a cone in $R^3_+(d)$ (resp. $R^3_-(d)$) then Corollary 3.3 and the results of [9] imply

$$\text{(5.11)}\quad \begin{aligned} E^\infty(u_f, C^\pm) &= c^2(\pm\infty)\, \rho(\pm\infty) \int_{C^\pm} (|p|^2 + q^2)\, |\hat{h}_-(p,q)|^2\, dpdq \\ &= \rho(\pm\infty) \int_{C^\pm} |\hat{f}_-(\omega_\pm(p,q),p,q)|^2\, dpdq. \end{aligned}$$

It follows that if

$$\text{(5.12)}\qquad S = \{(x,y) : d_1 \le y \le d_2\}$$

is a slab then

$$\text{(5.13)}\qquad E^\infty(u_f, S) = 0.$$

The Guided Component. Consider the family of cones defined by

$$\text{(5.14)}\qquad C(\varepsilon,d) = \{(x,y) : |y - d| < \varepsilon|x|\}$$

where $\varepsilon > 0$. Then, in contrast to (5.11), one has

$$\text{(5.15)}\qquad E^\infty(u_k, C(\varepsilon,d)) = E^\infty(u_k, R^3) = \|A^{1/2} h_k\|^2_{\mathcal{H}}$$

for every $\varepsilon > 0$ and $k = 1, 2, 3, \cdots$ (see [10, Theorem 5.5]). Finally, if S is the slab defined by (5.12) then one can show by means of Theorem 4.4 that

$$\text{(5.16)}\quad E^\infty(u_k, S) = \int_{\Omega_k} |\tilde{f}_k(\omega_k(|p|),p)|^2 \left\{\int_{d_1}^{d_2} \psi_k^2(y,p)\, c^{-2}(y)\, \rho^{-1}(y)dy\right\} dp.$$

BLANK PAGE

§6. Semi-Infinite and Finite Layers.

The preceding analysis is extended in this section to the cases of semi-infinite and finite layers of stratified fluid. The extensions are based on the normal mode expansions for these cases that were derived in [12, §9]. Only the principal concepts and results are formulated here since the proofs are entirely analogous to those of the preceding sections.

Semi-Infinite Layers. As in [12, §9] the fluid is assumed to occupy the domain R^3_+ and to satisfy the Dirichlet or Neumann boundary condition. Here the functions $\rho(y)$ and $c(y)$ are assumed to be Lebesgue measurable and satisfy

$$0 < \rho_m \le \rho(y) \le \rho_M < \infty, \quad 0 < c_m \le c(y) \le c_M < \infty \tag{6.1}$$

and

$$|\rho(y) - \rho(\infty)| \le C\, y^{-\alpha}, |c(y) - c(\infty)| \le C\, y^{-\alpha} \tag{6.2}$$

for all $y > 0$ where ρ_m, ρ_M, $\rho(\infty)$, c_m, c_M, $c(\infty)$, C and α are constants and

$$\alpha > 3/2. \tag{6.3}$$

As in [12, §9] the acoustic propagators for $\rho(y)$ and $c(y)$ corresponding to the Dirichlet and Neumann conditions will be denoted by A^0 and A^1, respectively. They are selfadjoint non-negative linear operators in $\mathcal{H}_+ = L_2(R^3_+, c^{-2}(y)\ \rho^{-1}(y)dxdy)$.

The normal mode functions $\psi^j(x,y,p,\lambda)$ for A^j, as defined in [12, §9], are parameterized by $(p,\lambda) \in \Omega = \{(p,\lambda) \mid \lambda > c^2(\infty)|p|^2\}$.

Their asymptotic form for $y \to \infty$ is given by

$$(6.4) \qquad \psi^j(x,y,p,q) \sim \frac{c(|p|,\lambda)}{2\pi}\left\{e^{i(p\cdot x-yq)} + R^j\, e^{i(p\cdot x+yq)}\right\}$$

where $q = q(|p|,\lambda) = (\lambda\, c^{-2}(\infty) - |p|^2)^{1/2}$, $c(|p|,\lambda) = (\rho(\infty)/4\pi q(|p|,\lambda))^{1/2}$, $R^j = R^j(|p|,\lambda)$ and $|R^j(|p|,\lambda)| = 1$. As in the preceding sections, it will be convenient to introduce new parameters: $(p,\lambda) \to (p,q)$ $= (p,q(|p|,\lambda)) \in R^3_+$ and normal mode functions

$$(6.5) \qquad \phi^j_+(x,y,p,q) = (2q)^{1/2}\, c(\infty)\, \psi^j(x,y,p,\lambda)$$

where

$$(6.6) \qquad \lambda = \lambda(p,q) = c^2(\infty)(|p|^2 + q^2).$$

The asymptotic form of ϕ^j_+ is

$$(6.7) \quad \phi^j_+(x,y,p,q) \sim \frac{c(\infty)\rho^{1/2}(\infty)}{(2\pi)^{3/2}}\left\{e^{i(p\cdot x-qy)} + R^j\, e^{i(p\cdot x+qy)}\right\}, \quad y \to \infty.$$

The second family ϕ^j_- defined by

$$(6.8) \qquad \phi^j_-(x,y,p,q) = \overline{\phi^j_+(x,y,-p,q)}$$

is also needed. It satisfies

$$(6.9) \quad \phi^j_-(x,y,p,q) \sim \frac{c(\infty)\rho^{1/2}(\infty)}{(2\pi)^{3/2}}\left\{e^{i(p\cdot x+qy)} + \overline{R^j}\, e^{i(p\cdot x-qy)}\right\}, \quad y \to \infty.$$

The expansion theorem of [12, §9] implies that the limits

$$(6.10) \quad \hat{f}^j_\pm(p,q) = L_2(R^3_+)\text{-}\lim_{M\to\infty} \int_0^M \int_{|x|\le M} \overline{\phi^j_\pm(x,y,p,q)}\, f(x,y)\, c^{-2}(y)\, \rho^{-1}(y)\,dx\,dy$$

exist. Moreover, if

(6.11) $$\Phi^j_\pm : \mathcal{H} \to L_2(R^3_+)$$

is defined by $\Phi^j_\pm f = \hat{f}^j_\pm$ then $\Phi^j_\pm$ is a partial isometry with range $L_2(R^3_+)$ and

(6.12) $$\Phi^{j*}_\pm \Phi^j_\pm + \sum_{k=1}^{N^j_0-1} \Psi^{j*}_k \Psi^j_k = 1$$

where $\Psi^j_k : \mathcal{H} \to L_2(\Omega_k)$ are the partial isometries associated with the guided wave normal modes $\psi^j_k(x,y,p)$ of [12, §9].

Normal mode expansions for A^j are given by (6.12) with either the + or - sign. (6.12) implies that the orthogonal projections in $\mathcal{H}_+$ defined by

(6.13) $$\begin{cases} P^j_f = \Phi^{j*}_+ \Phi^j_+ = \Phi^{j*}_- \Phi^j_- \\ \\ P^j_k = \Psi^{j*}_k \Psi^j_k, \ 1 \le k < N^j_0 \end{cases}$$

form a complete family that reduces A^j.

Transient Free Waves for Semi-Infinite Layers. The free component of a complex acoustic potential

(6.14) $$v(t,\cdot) = \exp\left(-i\,t\,(A^j)^{1/2}\right) h, \ h \in \mathcal{H}_+,$$

is given by

(6.15) $$v_f(t,\cdot) = P^j_f\, v(t,\cdot) = \exp\left(-i\,t\,(A^j)^{1/2}\right) P^j_f h$$

The Φ^j_--representation of v_f is

$$(6.16) \qquad v_f(t,x,y) = \int_{R^3_+} \phi^j_-(x,y,p,q) \exp(-i\, t\, \omega(p,q))\, \hat{h}^j_-(p,q)\, dp\, dq$$

where $\omega(p,q) = c(\infty)\sqrt{|p|^2 + q^2}$. Moreover, as in §3 one can write

$$(6.17) \qquad \phi^j_-(x,y,p,q) = \frac{c(\infty)\rho^{1/2}(\infty)}{(2\pi)^{3/2}} \left\{ e^{i(p\cdot x+qy)}\, I^j_-(y,p,q) + e^{i(p\cdot x-qy)} R^j_-(y,p,q) \right\}$$

where

$$(6.18) \qquad \begin{cases} \lim\limits_{y\to\infty} I^j_-(y,p,q) = 1, \\ \\ \lim\limits_{y\to\infty} R^j_-(y,p,q) = \overline{R^j(p,\lambda)}. \end{cases}$$

Then, proceeding as in §3, one can prove the following analogue of Theorem 3.1.

Theorem 6.1. For every $h \in \mathcal{H}_+$ let $v^0_f(t,\cdot)$ be defined by

$$(6.19) \qquad v^0_f(t,x,y) = \exp(-i\, t\, c(\infty)\, A_0^{1/2})\, h_0(x,y), \ (x,y) \in R^3_+$$

where $h_0 \in L_2(R^3)$ is the function whose Fourier transform is

$$(6.20) \qquad \hat{h}_0(p,q) = \begin{cases} c(\infty)\ \rho^{1/2}(\infty)\ \hat{h}^j_-(p,q), & (p,q) \in R^3_+, \\ \\ 0 & , (p,q) \in R^3_-. \end{cases}$$

Then

$$(6.21) \qquad \lim_{t\to\infty} \| v_f(t,\cdot) - v^0_f(t,\cdot) \|_{\mathcal{H}_+} = 0.$$

<u>Transient Guided Waves</u>. For both semi-infinite and finite layers the form of the guided components $v_k(t,\cdot)$ is precisely the same as for the case of an infinite layer. Thus the analysis of §4 applies unchanged to these cases.

BLANK PAGE

References

1. Dermenjian, Y., J. C. Guillot and C. H. Wilcox, Comportement asymptotique des solutions de l'équation des ondes associée à l'opérateur d'Epstein, C. R. Acad. Sci. Paris 287, 731-734 (1978).

2. Guillot, J. C., and C. H. Wilcox, Spectral analysis of the Epstein operator, Univ. Of Utah Technical Summary Rept. #27, 1975.

3. Guillot, J. C., and C. H. Wilcox, Spectral analysis of the Epstein operator, Proc. Roy. Soc. Edinburgh 80A, 85-98 (1978).

4. Kato, T., Perturbation Theory of Linear Operators, New York, Springer-Verlag, 1966.

5. Matsumura, M., Asymptotic behavior at infinity for Green's functions of first order systems with characteristics of nonuniform multiplicity, Publ. RIMS, Kyoto Univ. 12, 317-377 (1976).

6. Wilcox, C. H., Initial-boundary value problems for linear hyperbolic partial differential equations of the second order, Arch. Rational Mech. Anal. 10, 361-400 (1962).

7. Wilcox, C. H., Transient electromagnetic wave propagation along a dielectric-clad conducting plane, Univ. of Utah Technical Summary Rept. #22, May 1973.

8. Wilcox, C. H., Transient acoustic wave propagation in a symmetric Epstein duct, Univ. of Utah Technical Summary Rept. #25, May 1974.

9. Wilcox, C. H., Scattering Theory for the d'Alembert Equation in Exterior Domains, Lecture Notes in Mathematics, V. 442, Springer-Verlag 1975.

10. Wilcox, C. H., Transient electromagnetic wave propagation in a dielectric waveguide, Istitute Nazionale di Alta Matematica, Symposia Mathematica XVIII, 239-277 (1976).

11. Wilcox, C. H., Transient acoustic wave propagation in an Epstein duct, Univ. of Utah Technical Summary Rept. #36, November 1979.

12. Wilcox, C. H., Spectral analysis of sound propagation in stratified fluids, Univ. of Utah Technical Summary Rept. #38, April 1980.

Unclassified

SECURITY CLASSIFICATION OF THIS PAGE (When Data Entered)

REPORT DOCUMENTATION PAGE		READ INSTRUCTIONS BEFORE COMPLETING FORM
1. REPORT NUMBER TSR #42	2. GOVT ACCESSION NO. A104 787	3. RECIPIENT'S CATALOG NUMBER
4. TITLE (and Subtitle) (6) Transient Acoustic Wave Propagation in Stratified Fluids.		5. TYPE OF REPORT & PERIOD COVERED (9) Technical Summary Report,
		6. PERFORMING ORG. REPORT NUMBER (14) TSR-#42
7. AUTHOR(s) (10) Wilcox, Calvin H. /Wilcox		8. CONTRACT OR GRANT NUMBER(s) (15) N00014-76-C-0276
9. PERFORMING ORGANIZATION NAME AND ADDRESS Department of Mathematics University of Utah Salt Lake City, UT 84112		10. PROGRAM ELEMENT, PROJECT, TASK AREA & WORK UNIT NUMBERS Task No. NR-041-370
11. CONTROLLING OFFICE NAME AND ADDRESS Office of Naval Research, Code 432 Arlington, VA 22217		(11) 12. REPORT DATE July 1981
		13. NUMBER OF PAGES 47 (12) 49
14. MONITORING AGENCY NAME & ADDRESS (if different from Controlling Office)		15. SECURITY CLASS. (of this report) Unclassified
		15a. DECLASSIFICATION/DOWNGRADING SCHEDULE

16. DISTRIBUTION STATEMENT (of this Report)

Approved for public release, distribution unlimited

17. DISTRIBUTION STATEMENT (of the abstract entered in Block 20, if different from Report)

18. SUPPLEMENTARY NOTES

None

19. KEY WORDS (Continue on reverse side if necessary and identify by block number)

Sound propagation
Normal mode expansion
Acoustic duct
Plane-stratified fluid
Transient free wave
Transient guided wave

20. ABSTRACT (Continue on reverse side if necessary and identify by block number)

Abstract.

Transient acoustic wave propagation is analyzed for the case of plane-stratified fluids having density $\rho(y)$ and sound speed $c(y)$ at depth y. For infinite fluids it is assumed that the (in general discontinuous) functions $\rho(y)$, $c(y)$ are uniformly positive and bounded and satisfy

DD FORM 1 JAN 73 1473 EDITION OF 1 NOV 65 IS OBSOLETE
S/N 0102-LF-014-6601

Unclassified 401393

SECURITY CLASSIFICATION OF THIS PAGE (When Data Entered)

$$|\rho(y) - \rho(\pm\infty)| \le C(\pm y)^{-\alpha}, \quad |c(y) - c(\pm\infty)| \le C(\pm y)^{-\alpha}$$

for $\pm y > 0$, where $\alpha > 3/2$. Semi-infinite and finite layers are also treated. The acoustic potential is a solution of the wave equation

$$\frac{\partial^2 u}{\partial t^2} - c^2(y)\,\rho(y)\,\nabla \cdot (\rho^{-1}(y)\,\nabla u) = f(t,x,y)$$

where $x = (x_1, x_2)$ are horizontal coordinates and $f(t,x,y)$ characterizes the wave sources. The principal results of the analysis show that u is the sum of a free component, which behaves like a diverging spherical wave for large t, and a guided component which is approximately localized in regions $|y - y_j| < h_j$ where $c(y)$ has minima and propagates outward in horizontal planes like a diverging cylindrical wave.

LMED

-8